Int. J. Multiphase Flow Vol. 13, No. 5, pp. 717–722, 1987
Printed in Great Britain.

0301-9322/87 $3.00 + 0.00

BRIEF COMMUNICATION

AN INTERPRETATION FOR THE SOLID-PHASE PRESSURE IN SLOW, FLUID–PARTICLE FLOWS

R. C. GIVLER
Fluid and Thermal Sciences Department, Sandia National Laboratories, Albuquerque, NM 87185, U.S.A.

(*Received* 3 *January* 1986; *in revised form* 8 *March* 1987)

1. INTRODUCTION

Continuum mixture theories provide a convenient and general mathematical structure in which one can describe the rheological behavior of multiphase flows. The quintessential feature of this approach assumes that each material phase may be modeled as an interacting continuum. In the case of particulate suspensions, we associate the dispersed particles with a "fluid-like" continuum, herein referred to as a solid phase; the notion of a solid-phase continuum facilitates the construction of appropriate constitutive models. A quantity that has been difficult to conceptualize, and yet arises quite naturally from the mathematical formalisms in dealing with these flows, is an isotropic pressure assigned to the solid phase.

The notion of a solid-phase pressure has received considerable attention in the literature with regard to its physical meaning as well as its mathematical necessity. In many cases, the solid-phase pressure has been described in terms of the physics of the multiphase flow process. In section 2, we give a succinct description of the various models that have been used in the past to denote the solid-phase pressure. Even this partial listing serves to illustrate the diversity in which the idea of a solid-phase pressure has been conceived.

In the present communication, we seek a quantitative description for the solid-phase pressure during the slow, steady flow of a dilute, monodisperse suspension of spherical particles. We consider the macroscopic motion of the suspension to be dominated by viscous forces, however, we assume that the local inertia of the fluid in the vicinity of the particle will give rise to lift and "inertially corrected" drag forces. To be consistent, then, the expression for the solid-phase pressure must account for the local inertia of the fluid. In sum, we view the solid-phase pressure to be an average of the disturbance pressure in the vicinity of a single, spherical particle.

It can be shown that the disturbance pressure field surrounding a particle is due to the relative velocity between the particle and the fluid. We consider the effects of the local fluid inertia for this problem based upon the matched asymptotic analysis of Chester & Breach (1969). In section 3, we calculate the disturbance pressure for uniform, streaming flow past a sphere to obtain an insightful interpretation for the solid-phase pressure. Thus, a consistent definition of the solid-phase pressure is proposed for the class of two-phase flows considered here. The implications of this form for the solid-phase pressure, in view of the two-phase flow equations are discussed.

2. THE OCCURRENCE OF AN INTERFACE PRESSURE IN MIXTURE THEORIES

The behavior of multiphase material systems may be described mathematically by writing continuum balance laws of mass, momentum and energy for each constituent phase. The details of this particular approach in modeling multiphase flows may be found in the work of Drew & Segel (1971), Ishii (1975) and Passman *et al.* (1984). A significant asset of this approach is the general framework upon which a variety of two-phase flow models may be constructed. Classical mixture theories assume that the constituents are so intimately intermixed that all phases are ubiquitous in space. The discrete nature of the phases is maintained, however, by assigning separate

velocity and volume fraction fields to each phase. Within this mathematical structure each phase may interact with other phases and/or any physical boundaries. In the most general sense, this interaction may be in the form of momentum and energy exchanges between the phases. The precise nature of the phase interactions is provided via proper constitutive equations. In most cases, the interaction phenomena between the phases, for disperse, dilute mixtures, can be derived from local hydrodynamic analyses of single-particle motions. The drag force between phases, for example, is in part due to the Stokes resistance experienced, say, by a settling sphere. For the ensuing discussion we shall consider only isothermal, chemically inert mixtures such that the rheological behavior is derived by statements of conservation of mass and momentum for each phase. Furthermore, for definiteness we restrict our attention to disperse, two-phase systems in contrast to separated or transitory two-phase flows categorized by Ishii (1975).

For a saturated fluid–particle mixture, the momentum equation which governs the motion of the solid phase has been written by McTigue *et al.* (1986) as

$$\gamma_s \phi_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) = -\phi_s \nabla p_s + \mathbf{m}_s^+ + \gamma_s \phi_s \mathbf{b}_s - (p_s - p_f) \nabla \phi_s, \qquad [1]$$

where γ_s is the particle density, ϕ_s is the volume fraction of solid particles, $\mathbf{v}_s$ is the solid velocity, p_s and p_f are the phase pressures, $\mathbf{b}_s$ is the body force and $\mathbf{m}_s^+$ denotes the interaction forces of lift and drag acting between the phases.†

The last term in [1] describes a contribution to the momentum equation arising from pressure differences between the two phases.‡ This disparity of pressures has been attributed to the existence of an "interface pressure" and will be referred to as such hereafter. Written in this form, and assuming tentatively that $p_s > p_f$, it is apparent that the interface pressure is diffusive inasmuch as it can be shown that it leads to a homogenizing of the particle concentration field: local increases in particle concentration produce a momentum flux which opposes the concentration gradient. This feature of the continuum model is consistent with experimental observations by Segré & Silberberg (1962), who report diffuse particle concentrations across the tube diameter.

The model, two-phase mixture equations require an additional equation to formally close the equation set. The traditional approach has been to propose a constitutive equation among the nonkinematically related variables (i.e. p_f and p_s). Previous investigators, when confronted with the explanation of a solid-phase pressure and the associated proposition of a constitutive model, have resorted to a variety of approaches. We contend that any proposed interpretation of the interface pressure should be related ultimately to the physical nature of the particular two-phase system under consideration. We review some of the ideas advanced in the past to characterize the interface pressure.

The simplest assumption one could make concerning the closure equation for the phase pressure difference is to assume equal pressures. In this case, the model equations for the two-phase flow system retain only one independent pressure field. Stuhmiller (1977) has shown that this assumption leads to instabilities for initial-value problems in flows where the phase velocities are not equal. Of course, these flows are usually the ones of most interest. The instabilities are a result of complex characteristic roots imparted to the fundamental equation set by assuming equal pressures. The case of equal pressures between phases has also been observed to preclude unique solutions for nonuniform particle distributions in slow flows of dilute suspensions between parallel plates (McTigue *et al.* 1986). In particular, one will note from [1] that if $p_s = p_f$, the highest order derivative in the solid-phase volume fraction is lost. Furthermore, it can be shown that [1] becomes totally independent of the volume fraction under certain constitutive assumptions appropriate for dilute suspensions.§ Thus, questions of uniqueness of solution for the volume fraction arise.

To improve upon the constitutive equation for the interface pressure, one must consider the physical nature of the flow process. For the case of a particulate phase of spherical particles

†One interpretation of the interaction forces, $\mathbf{m}_s^+$, which include the effects of local fluid inertia, is given by McTigue *et al.* (1986). These forces acting on the dispersed phase have been generalized from analyses for the flow around a single particle.

‡It is easily shown that there is no corresponding contribution to the fluid-phase momentum equation from pressure differences between phases. Hence, we use only the solid-phase momentum equation for illustration.

§Specifically, $\mathbf{m}_s^+$ is a linear function of the volume fraction (McTigue *et al.* 1986).

suspended in a fluid, Stuhmiller (1977) and Prosperetti & Jones (1984) have investigated the local flow in the vicinity of an isolated sphere, where their interpretations of the interface pressure account for the local inertia of the flow. By considering the flow of an *inviscid* fluid around a sphere, Prosperetti & Jones (1984) have proposed the following form for the pressure difference:

$$p_s - p_f = -\frac{1}{4}\gamma_f(\mathbf{v}_f - \mathbf{v}_s)^2. \quad [2]$$

The interesting feature here is that the solid-phase pressure due to the presence of the particles is less than the bulk fluid pressure. It can be shown that this assumption will lead one to calculate particle concentration profiles which contradict experimental observations for 2-D, Poiseuille flow of a dilute suspension. In particular, one may note from a manipulation of [1] that the pressure difference given here by [2] leads to a force which produces a particle flux in the direction of increasing particle concentration gradient. This result is both antidiffusive and nonphysical.

When the suspended particles are small enough for Brownian motion to become important, Drew (1976) and Nunziato (1983) have shown that the interface presure is diffusive and related to the derivative of a thermodynamic potential. These ideas are consistent with the notion of a "Brownian stress", as defined by Batchelor (1976). The random motion of the particles due to collisions with the fluid molecules gives rise to a diffusive force which acts to homogenize particle concentrations. In this instance, the interface pressure is given by the expression

$$p_s - p_f \sim \frac{3\kappa\zeta}{4\pi a^3}, \quad [3]$$

where κ is Boltzmann's constant, ζ is the absolute temperature and a is the particle radius. It is noted from [3] that, for two-phase systems consisting of small particles, the interface pressure increases, respectively, with increasing temperature and decreasing particle size.

As a final example, when the dispersed, solid-phase concentration exceeds the dilute limit, particle–particle interactions become important, say, in the case of fluidized beds. To model these flows with multiphase mixture theories, several researchers have either defined (Homsy *et al.* 1980), or implied tacitly (Anderson & Jackson 1967), the concept of a "collision" pressure to account for the elastic interactions between the particles. These effects are significant if the bed is less than fully fluidized. In other related work (Stiehadieh *et al.* 1984), we find similar expressions for the interface pressure which attempt to model particle contact forces for nondilute concentrations of suspensions.

In the cases cited here and others as well, the constitutive equations for the interface pressure have been postulated based on the physics of the flow. Many of these models embody the interaction physics of a single particle with the suspending fluid. We adopt a similar strategy in the next section to offer an explanation for the interface pressure evolving from the slow flow of a dilute suspension of spherical particles.

3. THE FLOW FIELD IN THE VICINITY OF AN ISOLATED SPHERE

For the case of a dilute suspension of monodisperse, spherical particles undergoing slow, steady flow, we propose a formula for and offer physical insight toward the notion of the solid-phase pressure. Inertial effects of the macroscopic motion of the suspension are neglected, however, the deliberate indication of "slow flow" implies that local fluid inertia is important on a length scale compared to the particle diameter. It can be shown (Ho & Leal 1974) that local inertial effects are responsible for lateral particle migrations in bounded shearing flows of the type considered here. Our motivation emanates from the need to develop a suitable expression for the interface pressure in processing-type flows of dilute suspensions.

If the suspension is dilute, as assumed, we cannot elicit any contribution to the solid-phase pressure from particle interaction forces. Therefore, we postulate that the solid-phase pressure must be due to the local disturbance field created by the relative motion between a single particle and the fluid. Intuition suggests (Nunziato 1983) that for the class of flows considered herein the functional form of the pressure difference must include the effect of local fluid inertia, such that

$$p_s - p_f = c_0\gamma_f(\mathbf{v}_f - \mathbf{v}_s)^2, \quad [4]$$

where c_0 is a constant, γ_f is the mass density of the fluid and $(\mathbf{v}_f - \mathbf{v}_s)$ is the difference between the velocity of the undisturbed fluid motion and that of the particle. Relative motion between the dispersed particulate phase and the suspending fluid can be due to density differences between the constituents, as given by Stokes settling, the correction to the particle drag force given by Faxén (1922) and/or lift forces which act transversely to the undisturbed fluid streamlines. Here, we seek to justify mathematically the functional form given by [4] as well as identify the coefficient c_0. Toward this end, we seek to calculate the average disturbance pressure field due to uniform flow past a spherical particle.†

This problem has received widespread attention in the literature and we cite for our purpose here some results of earlier work. A consistent approximation to the Navier–Stokes equations for the slow flow around a sphere, for small Reynolds numbers, Re, has been given initially by Proudman & Pearson (1957) and later extended by Chester & Breach (1969). Here, Re is given by $\gamma_f a U/\mu_f$, based upon the particle radius, a, and the uniform streaming velocity, U. Each of these investigations used the method of matched asymptotic expansions to obtain a solution for the flow field which is uniformly valid both near and far from the particle.

Of specific relevance for our calculations of the interface pressure is a description of the local fluid pressure on the surface of the spherical particle. This relation is given in the paper by Chester & Breach (1969, [6.7]), and is written here for convenience. The reader is directed to the original paper for the details leading to the derivation of the equation,

$$p - p_f = -\int^{\theta} \left(\frac{\partial^3 \psi}{\partial r^3}\right)_{r=1} \frac{d\theta}{\sin\theta}, \qquad [5]$$

where ψ is the normalized stream function of the inner expansion and r is the normalized radial coordinate. The pressure difference in [5] has been normalized by the quantity $\mu_f U/a$ and vanishes far from the body. Through terms of order Re,

$$\psi = \psi_0 + \mathrm{Re}\,\psi_1 + \ldots, \qquad [6]$$

$$\psi_0 = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta \qquad [7]$$

and

$$\psi_1 = \frac{3}{32}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta - \frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)\cos\theta\,\sin^2\theta. \qquad [8]$$

The origin of the spherical polar coordinate system (r, θ, ϕ), is fixed at the center of the particle and the polar angle, θ, is measured anticlockwise with respect to the uniform streaming fluid velocity. From [5], we obtain

$$p - p_f = \left[-\frac{3}{2}\left(1 + \frac{3}{8}\mathrm{Re}\right)\cos\theta + \frac{27}{32}\mathrm{Re}\cos^2\theta\right] + O(\mathrm{Re}^2 \log \mathrm{Re}). \qquad [9]$$

The first two terms on the r.h.s. of [9] coincide with the pressure field calculated from the Stokes expansion, while the third term accounts for the influence of inertia which finds its way into the inner expansion through the process of matching near and far field solutions in the overlap region.

We define the solid-phase pressure in the continuum sense to be, in part, the *average* pressure experienced by a single particle and derived from the local disturbance pressure field. It follows that this inertial correction may be obtained by integrating [9] over the particle surface, Ω,

$$p_s - p_f = \frac{1}{\Omega}\int (p - p_f)\, d\Omega = \frac{9}{32}\mathrm{Re} + O(\mathrm{Re}^2). \qquad [10]$$

It is interesting to note that while the first two terms on the r.h.s. of [9] contribute to the drag they do not contribute to the integral of the fluid pressure over the particle surface given by [10]. It is the third term in [9], which is an even function in θ, that contributes to the interfacial pressure through terms of order Re. This term is the first correction to the flow field close to the particle

†One could alternatively consider the case of a slow, uniform translation of the sphere through a quiescent, viscous liquid. The solution of this related problem differs from the one considered here only by a uniform streaming motion of the fluid. The local disturbance pressure field enveloping the body is the same for either case.

from inertia. The next correction to the interface pressure, as shown in [10], will be order Re^2 since the intermediate term Re^2 log Re from the analysis by Chester & Breach (1969) does contribute to the drag but not to the averaged interface pressure.

Finally, from [4] and [10] we may write the dimensional form for the solid-phase pressure as

$$p_s = p_f + \frac{9}{32}\gamma_f(\mathbf{v}_f - \mathbf{v}_s)^2. \quad [11]$$

4. CONCLUSIONS

We have shown that, for the slow flow of a dilute suspension, one contribution to the pressure assigned to the dispersed particles may be calculated based upon the local fluid inertia. The interpretation given here for the solid-phase pressure is derived from the local disturbance pressure field in the suspending fluid due to the relative motion between the particles and the fluid. In the case of zero relative motion between the phases, the inertial correction to the solid-phase pressure vanishes. We note that this interpretation of the solid-phase pressure is intimately associated with the flow process and is therefore physically motivated.

The solid-phase pressure, as shown in this paper, is a dynamic effect and exceeds the fluid-phase pressure by an inertial correction. The interface pressure, $p_s - p_f$, is diffusive in nature inasmuch as local fluid inertia will homogenize particle concentration distributions. This follows from the fact that $p_s > p_f$, which produces a solid momentum flux in the direction opposite to increasing concentration gradients. Therefore, a diffusive interface pressure is one mechanism by which particle concentrations become more homogeneous in flows where the local fluid inertia is important.

Acknowledgement—This work was supported by the U.S. Department of Energy under Contract No. DE-AC04-76DP00789; their support is gratefully acknowledged.

REFERENCES

ANDERSON, T. B. & JACKSON, R. 1967 A fluid mechanical description of fluidized beds. *Ind. Engng Chem. Fundam.* **6,** 527–538.

BATCHELOR, G. K. 1976 Brownian diffusion of particles with hydrodynamic interaction. *J. Fluid Mech.* **74,** 1–29.

CHESTER, W. & BREACH, D. R. 1969 On the flow past a sphere at low Reynolds number. *J. Fluid Mech.* **37,** 751–760.

DREW, D. A. 1976 Two-phase flows: constitutive equations for lift and Brownian motion and some basic flows. *Archs ration. Mech. Analysis* **62,** 149–163.

DREW, D. A. & SEGEL, L. A. 1971 Averaged equations for two-phase flows. *Stud. appl. Math.* **L,** 205–231.

FAXÉN, H. 1922 Die Bewegung einer starren Kugel längs der Achse eines mit zäher Flüssigkeit gefüllten Rohres. *Ark. Mat. Astr. Fys.* **17,** 1–28.

HO, B. P. & LEAL, L. G. 1974 Inertial migration of rigid spheres in two-dimensional unidirectional flows. *J. Fluid Mech.* **65,** 365–400.

HOMSY, G. M., EL-KAISSY, M. M. & DIDWANIA, A. 1980 Instability waves and the origin of bubbles in fluidized beds—II. *Int. J. Multiphase Flow* **6,** 305–318.

ISHII, M. 1975 *Thermo-fluid Dynamic Theory of Two-Phase Flow.* Eyrolles, Paris.

MCTIGUE, D. F., GIVLER, R. C. & NUNZIATO, J. W. 1986 Rheological effects of non-uniform particle distributions in dilute suspensions. *J. Rheol.* **30,** 1053–1076.

NUNZIATO, J. W. 1983 A multiphase mixture theory for fluid–particle flows. In *Theory of Dispersed Multiphase Flow* (Edited by MEYER, R. E.), pp. 191–226. Academic Press, New York.

PASSMAN, S. L., NUNZIATO, J. W. & WALSH, E. K. 1984 A theory for multiphase mixtures. In *Rational Thermodynamics,* 2nd edn (Edited by TRUESDELL, C.), pp. 286–325. Springer, New York.

PROSPERETTI, A. & JONES, A. V. 1984 Pressure forces in disperse two-phase flow. *Int. J. Multiphase Flow* **10,** 425–440.

PROUDMAN, I. & PEARSON, J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* **2,** 237–262.

SEGRÉ, G. & SILBERBERG, A. 1962 Behavior of macroscopic rigid spheres in Poiseuille flow. *J. Fluid Mech.* **14,** 136–157.

STIEHADIEH, B., GIDASPOW, D. & LYCZKOWSKI, R. W. 1984 Hydrodynamics of fluidization in a semicircular bed with a jet. *AIChE Jl* **30,** 529–536.

STUHMILLER, J. H. 1977 The influence of interfacial pressure forces on the character of two-phase flow model equations. *Int. J. Multiphase Flow* **3,** 551–560.